§ t-structure と hearts (復習)

Def 7.1

$\mathcal{D}$: 三角圏.

$\mathcal{D}$ の充満部分圏 $\mathcal{D}^{\leq 0}$ と $\mathcal{D}^{\geq 0}$ が次の条件をみたすとき,
$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ は $\mathcal{D}$ の t-構造 (t-structure) という.

またこのとき,

$$\begin{cases} \mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n] \\ \mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n] \end{cases}$$

と定める.

(条件)

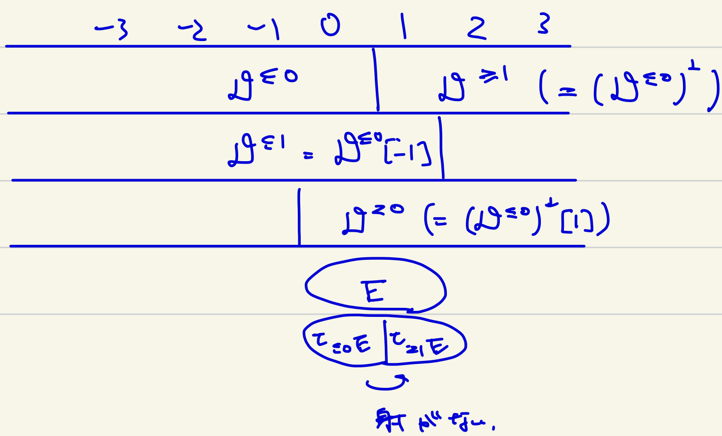
① $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$, $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ が成立.

② $\mathcal{D}^{\geq 1} \subset (\mathcal{D}^{\leq 0})^{\perp}$ が成立.

③ $\forall E \in \mathcal{D}$, $\exists$ exact tr.

$$\tau_{\leq 0}E \to E \to \tau_{\geq 1}E \to (\tau_{\leq 0}E)[1]$$

$(\tau_{\leq 0}E \in \mathcal{D}^{\leq 0}$, $\tau_{\geq 1}E \in \mathcal{D}^{\geq 1})$



$(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ が t-structure を与えるとき、$\mathcal{D}$ の部分圏

$$\mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$$

を、t-構造の核 (the heart of t-structure) とよぶ。

<u>Fact</u> (GM03, Thm IV.4.4, HTT (山本他) Thm 8.1.9 (i))

t-structure の heart は Abel 圏の構造を持つ

<u>例</u>

Abel 圏 A の導来圏 $\mathcal{D} := \mathcal{D}(A)$ に対し、

$$\mathcal{D}^{\le 0} := \{ E \in \mathcal{D} \mid \mathcal{H}^i(E) = 0, \quad i > 0 \}$$

$$\mathcal{D}^{\ge 0} := \{ E \in \mathcal{D} \mid \mathcal{H}^i(E) = 0, \quad i \le -1 \}$$

と定めると、$(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ は $\mathcal{D}$ の t-structure で、

$$A \xrightarrow{\sim} \mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0} \quad ; \quad E \mapsto (\cdots \to 0 \to E \to 0 \to \cdots)$$

<u>Def 7.6.</u>

三角圏 $\mathcal{D}$ の t-structure $\mathcal{D}^{\le 0} \subset \mathcal{D}$ が有界であるとは、

$$D = \bigcup_{i,j \in \mathbb{Z}} (\mathcal{D}^{\le i} \cap \mathcal{D}^{\ge j})$$

が成立すること。

Prop

自然な包含函手 $\iota: \mathcal{D}^{\leq n} \longrightarrow \mathcal{D}$ を考える。
このとき、函手 $\tau_{\leq n}: \mathcal{D} \to \mathcal{D}^{\leq n}$ が存在して、

$$\mathrm{Hom}_{\mathcal{D}^{\leq n}}(Y, \tau_{\leq n}(X)) \simeq \mathrm{Hom}_{\mathcal{D}}(\iota(Y), X).$$

同様に、自然な包含函手 $\iota': \mathcal{D}^{\geq n} \longrightarrow \mathcal{D}$ に対し、
函手 $\tau_{\geq n}: \mathcal{D} \longrightarrow \mathcal{D}^{\geq n}$ が存在して、

$$\mathrm{Hom}_{\mathcal{D}^{\geq n}}(\tau_{\geq n}(X), Y) \simeq \mathrm{Hom}_{\mathcal{D}}(X, \iota'(Y)).$$

(proof: たとえば HTT (D-お.王?) prop 8.6を)

t-structure の heart $\mathcal{A} \subset \mathcal{D}$ が与えられたとき、
各 $i \in \mathbb{Z}$ に対して次の cohomological functor

$$\begin{array}{cccc} \mathcal{H}^i_{\mathcal{A}}: & \mathcal{D} & \longrightarrow & \mathcal{A} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & E & \longmapsto & \tau_{\geq i}\tau_{\leq i}(E)[i] \end{array}$$

が定義できる　$(\mathcal{H}^i_{\mathcal{A}}(E) = \tau_{\geq i}\tau_{\leq i}(E)[i])$

問7.7.

(i) $\mathcal{A}$: 三角圏 $\mathcal{D}$ の 加法部分圏.

このとき. $\mathcal{A}$ が 有界な t-structure の heart であることと.

次の 2つの 条件 が 成立すること. は 同値.

(a) $\forall k_1 > k_2 \ (\in \mathbb{Z})$, $\forall A, B \in \mathcal{A}$.

$$\mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0 .$$

(b) $0 \neq \forall E \in \mathcal{D}$, $\exists$ 整数列 $k_1 > k_2 > \cdots > k_n$,

$\exists$ exact tr.

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul] \arrow[u, phantom, "\triangledown"] & & & & A_n \arrow[ul] \arrow[u, phantom, "\triangledown"] &
\end{tikzcd} \quad -(*)$$

s.t. $\forall j$, $A_j \in \mathcal{A}[k_j]$

(ii) 三角圏 $\mathcal{D}$ が 有界な t-structure を持つとし.

その heart を $\mathcal{A}$ とおく.

このとき.

$$K(\mathcal{D}) \cong K(\mathcal{A}).$$

Lem

1つ目の条件(A)の下で、2つ目の完全三角の列は(同型を除いて)一意に定まる.

∵

$0 \neq E \in \mathcal{D}$ に対して、次の2つの列が得られたとする.

$\exists\ k_1 > k_2 > \cdots > k_n$

$$\exists\ \begin{tikzcd} 0 \arrow[rr] && E_1 \arrow[rr] \arrow[dl] && E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] && E_n = E \arrow[dl] \\ & A_1 \arrow[ul] && A_2 \arrow[ul] &&&& A_n \arrow[ul] \end{tikzcd}$$

s.t. $A_1 \in \mathcal{A}[k_1]$, $A_2 \in \mathcal{A}[k_2]$, $\ldots$, $A_n \in \mathcal{A}[k_n]$

$\exists\ k'_1 > k'_2 > \cdots > k'_m$

$$\exists\ \begin{tikzcd} 0 \arrow[rr] && E'_1 \arrow[rr] \arrow[dl] && E'_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E'_{m-1} \arrow[rr] && E'_m = E \arrow[dl] \\ & A'_1 \arrow[ul] && A'_2 \arrow[ul] &&&& A'_m \arrow[ul] \end{tikzcd}$$

s.t. $A'_1 \in \mathcal{A}[k'_1]$, $A'_2 \in \mathcal{A}[k'_2]$, $\ldots$, $A'_m \in \mathcal{A}[k'_m]$

Step 1.

$k'_1 \geq k_1$ として一般性を失わない.

まず、$E'_1 \to E'_2 \to \cdots \to E'_m \simeq E$ を合成して

射 $f'_1 : E'_1 \to E$ を得る.

(各 $A'_i \neq 0$ としてよいので、$f'_1 \neq 0$)

$$
\begin{tikzcd}
 & E'_1 \simeq A'_1 \in \mathcal{A}[k_1] \arrow[dl] \arrow[d, "f'_1"] \arrow[dr] & \\
E_{n-1} \arrow[r] & E \arrow[r] & A_n \in \mathcal{A}[k_n]
\end{tikzcd}
$$

$k'_1 \geqq k_1 > k_n$ より $\mathrm{Hom}(E'_1, A_n) = 0$.

$$
\begin{tikzcd}
 & E'_1 \arrow[dl, "s'_{n-1}"'] \arrow[d, "f'_1"] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] \\
E_{n-1} \arrow[r] & E
\end{tikzcd}
$$

よって $\mathrm{Hom}(E'_1, E_{n-1}) \to \mathrm{Hom}(E'_1, E)$ は全射で、

が可換になるような非自明な射 $s'_{n-1} : E'_1 \to E_{n-1}$ が存在.

このような操作をくり返して、非自明な射 $s'_1 : E'_1 \to E_1$ を得る.

$\mathcal{A}[k'_1] \ni A'_1 \simeq E'_1 \xrightarrow{s'_1} E_1 \simeq A_1 \in \mathcal{A}[k_1]$ と条件(a)より $k'_1 \leqq k_1$ $(\therefore k_1 = k'_1)$

$k_1 \geqq k'_1$ として同様の議論により、非自明な射 $s_1 : E_1 \to E'_1$ であって

$$
\begin{tikzcd}
 & E_1 \arrow[dl, "s_1"'] \arrow[d, "f_1"] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] \\
E'_1 \arrow[r, "f'_1"'] & E
\end{tikzcd}
$$

が可換となるものが存在.

$$
\begin{tikzcd}
E_1 \arrow[r, "f_1"] \arrow[d, "s_1"'] \arrow[dd, bend right=60, "\mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f_1) \arrow[r, "+1"] \arrow[d, "\exists"] & {} \\
E'_1 \arrow[r, "f'_1"] \arrow[d, "s'_1"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f'_1) \arrow[r] \arrow[d, "\exists"] & {} \\
E_1 \arrow[r] & E \arrow[r] & \mathrm{Cone}(f_1) \arrow[r] & {}
\end{tikzcd}
$$

(射の一意性：正しいがまだ示せてない)

図式の可換性から $s'_1 \circ s_1 = \mathrm{id}$. 同様に $s_1 \circ s'_1 = \mathrm{id}$.

ゆえに、$E_1 \simeq E'_1$ $(A_1 \simeq A'_1)$.

Step 2.

E_ℓ まで同型だとする。（つまり、$E_j \simeq E'_j$, $A_j \simeq A'_j$, $k_j = k'_j$, $1 \leq j \leq \ell$）

$k'_{\ell+1} \geqq k_{\ell+1}$ を仮定。

$$E'_{\ell+1} \to E'_{\ell+2} \to \cdots \to E'_m \simeq E$$

を合成して、非自明な射 $f'_{\ell+1} : E'_{\ell+1} \to E$ を得る。

図式

$$\begin{tikzcd}
E'_\ell \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & E'_{\ell+1} \arrow[r] \arrow[d, "f'_{\ell+1}"] \arrow[dr, phantom, "\circlearrowleft"] & A'_{\ell+1} \arrow[r, "+1"] \arrow[d, dashed, "\exists"'] & {} \\
E_\ell \arrow[r, "f_\ell"] & E \arrow[r] & \mathrm{Cone}(f_\ell) \arrow[r, "+1"] & {}
\end{tikzcd}$$

と、三角圏の公理3により $A'_{\ell+1} \to \mathrm{Cone}(f_\ell)$ を得る。

合成射 $h_{n-1} : E_\ell \to E_{n-1}$ において、$B_{n-1} := \mathrm{Cone}(h_{n-1})$ とおく。

$$\left\{\begin{array}{l}
E_\ell \to E_{n-1} \to B_{n-1} \xrightarrow{+1} \\
E_{n-1} \to E \to A_n \xrightarrow{+1} \\
E_\ell \xrightarrow{f_\ell} E \to \mathrm{Cone}(f_\ell) \xrightarrow{+1}
\end{array}\right.$$

に、八面体公理を用いて、dist. tri.

$$B_{n-1} \to \mathrm{Cone}(f_\ell) \to A_n \xrightarrow{+1}$$

を得る。

$$\begin{tikzcd}
& A'_{\ell+1} \arrow[dl] \arrow[d] \arrow[dr] & \in \mathcal{A}[k'_{\ell+1}] \\
B_{n-1} \arrow[r] & \mathrm{Cone}(f_\ell) \arrow[r] & A_n \in \mathcal{A}[k_n]
\end{tikzcd}$$

$k'_{\ell+1} \geq k_{\ell+1} > k_n$ より $\mathrm{Hom}(A'_{\ell+1}, A_n) = 0$.

よって、非自明な射 $A'_{\ell+1} \to B_{n-1}$ であって、図式が可換になるものが存在。

$$\begin{tikzcd} & A'_{\ell+1} \arrow[dl, "a"'] \arrow[d] \\ B_{n-1} \arrow[r] & \mathrm{Cone}(f_\ell) \end{tikzcd}$$

このような操作を繰り返すことで、非自明な射

$$A'_{\ell+1} \to A_{\ell+1}$$

を得る。条件 (a) より $k'_{\ell+1} \leq k_{\ell+1}$ $\quad$ ($\therefore$ $k_{\ell+1} = k'_{\ell+1}$).

また、

$$\begin{tikzcd} E'_\ell \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & E'_{\ell+1} \arrow[r] \arrow[d, dashed, "\exists"'] \arrow[dr, phantom, "\circlearrowright"] & A'_{\ell+1} \arrow[r, "+1"] \arrow[d] & {} \\ E_\ell \arrow[r] & E_{\ell+1} \arrow[r] & A_{\ell+1} \arrow[r, "+1"] & {} \end{tikzcd}$$

公理3より 射 $g'_{\ell+1} : E'_{\ell+1} \to E_{\ell+1}$ を得る。 $\quad$ ($g'_{\ell+1} \neq 0$)

$k_{\ell+1} \geq k'_{\ell+1}$ として 同様の議論で 射 $g_{\ell+1} : E_{\ell+1} \to E'_{\ell+1}$ を得る。

$$\begin{tikzcd} E_{\ell+1} \arrow[r, "f_{\ell+1}"] \arrow[d, "g_{\ell+1}"'] \arrow[dr, phantom, "\circlearrowright"] \arrow[dd, bend right=60, "\mathrm{id}"'] & E \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f_{\ell+1}) \arrow[r, "+1"] \arrow[d, "\exists"] & {} \\ E'_{\ell+1} \arrow[r, "f'_{\ell+1}"] \arrow[d, "g'_{\ell+1}"'] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f'_{\ell+1}) \arrow[r, "+1"] \arrow[d, "\exists"] & {} \\ E_{\ell+1} \arrow[r, "f_{\ell+1}"] & E \arrow[r] & \mathrm{Cone}(f_{\ell+1}) \arrow[r, "+1"] & {} \end{tikzcd}$$

(射の一意性に注意)

図式の可換性から $g'_{\ell+1} \circ g_{\ell+1} = \mathrm{id}$. $\quad$ 同様に $g_{\ell+1} \circ g'_{\ell+1} = \mathrm{id}$.

$\therefore$ $E_{\ell+1} \simeq E'_{\ell+1}$ $\quad$ したがって $A_{\ell+1} \simeq A'_{\ell+1}$ $\quad$ □

[Proof of 命7.7 (i)]

[$\Rightarrow$] $\mathcal{A}$ を有界な t-structure の heart とする.

Claim 1.

$^\forall k_1 > k_2 \ (\in \mathbb{Z}), \quad ^\forall A, B \in \mathcal{A},$

$\mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0$

∵

$$A[k_1] \in \mathcal{A}[k_1] = (\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0})[k_1] = \mathcal{D}^{\leq -k_1} \cap \mathcal{D}^{\geq -k_1}$$

$$B[k_2] \in \mathcal{D}^{\leq -k_2} \cap \mathcal{D}^{\geq -k_2}$$

$k_1 > k_2$ より $-k_1 < -k_2$ で, また t-structure の定義から

$\mathcal{D}^{\geq 1} \subset (\mathcal{D}^{\leq 0})^{\perp}$ なので $\mathcal{D}^{\geq -k_2} \subset (\mathcal{D}^{\leq -k_1})^{\perp}$

$\therefore \ \mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0.$

Claim 2.

$0 \neq {}^\forall E \in \mathcal{D}, \quad \exists k_1 > k_2 > \cdots > k_n \quad (\in \mathbb{Z})$

$$\exists \quad \begin{tikzcd} 0 = E_0 \arrow[rr] & & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, "{[1]}"] \arrow[u, phantom, "\triangle"] & & & & A_n \arrow[ul, "{[1]}"] \arrow[u, phantom, "\triangle"] & \end{tikzcd}$$

s.t. $\quad ^\forall j, \quad A_j \in \mathcal{A}[k_j]$

∵ 有界性から $\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} (\mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq j})$ なので,

$^\forall E \in \mathcal{D}, \quad \exists j \leq i \ (\in \mathbb{Z})$ s.t. $E \in \mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq j}.$

このとき、次の列が存在する！

$$
\begin{tikzcd}[column sep=small]
0 = \tau_{\leq j-1}(E) \arrow[rr] & & \tau_{\leq j}(E) \arrow[rr] \arrow[dl] & & \tau_{\leq j+1}(E) \arrow[r] \arrow[dl] & \cdots \arrow[r] & \tau_{\leq i-1}(E) \arrow[rr] & & \tau_{\leq i}(E) = E \arrow[dl] \\
& B_j \arrow[ul] \arrow[ur, phantom, "\triangledown"] & & B_{j+1} \arrow[ul] \arrow[ur, phantom, "\triangledown"] & & & & B_i \arrow[ul] \arrow[ur, phantom, "\triangledown"] &
\end{tikzcd}
$$

（各 B_k は、 $B_k = \mathcal{H}^k_{\mathcal{A}}(E)[-k] \in \mathcal{A}[-k]$ をみたす。）

$A_1 := B_j$, $A_2 := B_{j+1}$, $\cdots$, $A_{i-j+1} = B_j$ とおけば、

$$
\begin{array}{ll}
A_1 \in \mathcal{A}[-j], & k_1 = -j \\
 & \quad\ \vee \\
A_2 \in \mathcal{A}[-j-1], & k_2 = -j-1 \\
 & \quad\ \vee \\
\quad \vdots & \quad \vdots \\
 & \quad\ \vee \\
A_{i-j+1} \in \mathcal{A}[-i] & k_{i-j+1} = -i
\end{array}
$$

[$\Leftarrow$]

$$
\left\{
\begin{array}{l}
\mathcal{D}^{\leq 0} := \left\{ E \in \mathcal{D} \;\middle|\; \begin{array}{l} \exists!\, k_1 > k_2 > \cdots > k_m \geq 0 \\ \exists!\, \begin{tikzcd}[column sep=small] 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_m = E \arrow[dl] \\ A_1 \arrow[u] & & A_m \arrow[u] & \end{tikzcd} \\ \text{s.t. } \forall j,\ A_j \in \mathcal{A}[k_j]. \end{array} \right\} \\[3em]
\mathcal{D}^{\geq 0} := \left\{ E \in \mathcal{D} \;\middle|\; \begin{array}{l} \exists!\, 0 \geq k_1 > k_2 > \cdots > k_m \\ \exists!\, \begin{tikzcd}[column sep=small] 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_m = E \arrow[dl] \\ A_1 \arrow[u] & & A_m \arrow[u] & \end{tikzcd} \\ \text{s.t. } \forall j,\ A_j \in \mathcal{A}[k_j] \end{array} \right\}
\end{array}
\right.
$$

← (an より) 同型を除いて一意

$$
\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \left\{ E \in \mathcal{D} \;\middle|\; \begin{tikzcd}[column sep=small] 0 \arrow[r] & E \arrow[dl] \\ A \arrow[u, "+1"] \arrow[ur, phantom, "\triangledown"] & \end{tikzcd},\ A \in \mathcal{A} \right\} \cong \mathcal{A}.
$$

以下、$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ が bounded t-structure であることを示す。

Claim 1

$$\mathcal{D}^{\le -1} = \mathcal{D}^{\le 0}[1] \subset \mathcal{D}^{\le 0} \qquad (\Leftrightarrow \ \mathcal{D}^{\le 0} \subset \mathcal{D}^{\le 1})$$

$$\mathcal{D}^{\ge 0}[1] = \mathcal{D}^{\ge -1} \subset \mathcal{D}^{\ge 0} \qquad (\Leftrightarrow \ \mathcal{D}^{\ge 1} \subset \mathcal{D}^{\ge 0})$$

(∵)

$0 \ne E \in \mathcal{D}^{\le 0}$ をとると、定め方から

$$\exists \quad \begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul] \arrow[u, phantom, "\triangledown"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & & & A_n \arrow[ul] \arrow[u, phantom, "\triangledown"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
& \mathcal{A}[k_1] & & & & \mathcal{A}[k_n] &
\end{tikzcd}$$

s.t. $k_1 > k_2 > \cdots > k_n \geq 0$

$E[1]$ を考えると、

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & E_{i-1}[1] \arrow[rr] & & E_i[1] \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1}[1] \arrow[dl] \arrow[rr] & & E_n[1] = E[1] \arrow[dl] \\
& & A_i[1] =: A_i' \arrow[ul] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & A_{n-1}[1] =: A_{n-1}' \arrow[ul] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & A_n[1] =: A_n' \arrow[ul] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
& & \mathcal{A}[k_i+1] =: \mathcal{A}[k_i'] & & \mathcal{A}[k_{n-1}+1] =: \mathcal{A}[k_{n-1}'] & & \mathcal{A}[k_n+1] =: \mathcal{A}[k_n'] &
\end{tikzcd}$$

$A_i' \ne 0$ のとき、$k_i' = k_i + 1 \geq 1 > 0$ $\therefore$ $E[1] \in \mathcal{D}^{\le 0}$

$E \in \mathcal{D}^{\le 0}$ は任意にとれるので、$\mathcal{D}^{\le 0}[1] \subset \mathcal{D}^{\le 0}$. 他も同様。 ◢

Rem

$$\mathcal{D}^{\le -1} = \left\{ E \in \mathcal{D} \;\middle|\; \begin{array}{c} \exists!\, k_1 > k_2 > \cdots > k_m \geq 1 \\ \exists!\ \begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1 \arrow[ul] & & & & A_m \arrow[ul] &
\end{tikzcd} \\ \text{s.t. } \forall j,\ A_j \in \mathcal{A}[k_j]. \end{array} \right\}$$

$$\mathcal{D}^{\leq n} = \left\{ E \in \mathcal{D} \;\middle|\; \begin{array}{c} \exists!\, k_1 > k_2 > \cdots > k_m \geq -n \\ \exists!\, 0 = E_0 \to E_1 \to \cdots \to E_m = E \\ \quad A_1 \qquad\qquad A_m \\ \text{s.t. } \forall j,\ A_j \in \mathcal{A}[k_j]. \end{array} \right\}$$

Claim 2

$\mathcal{D}^{\geq 1} \subset (\mathcal{D}^{\leq 0})^{\perp}$

∵ 次を思い出す.

Lem 2.13

任意の完全三角 $E \to F \to G \xrightarrow{+1} E[1]$ in $\mathcal{D}$ と $M \in \mathcal{D}$ に対し,

$\mathrm{Hom}_{\mathcal{D}}(M, E) \to \mathrm{Hom}_{\mathcal{D}}(M, F) \to \mathrm{Hom}_{\mathcal{D}}(M, G)$ は完全.

$(\mathrm{Hom}_{\mathcal{D}}(G, M) \to \mathrm{Hom}_{\mathcal{D}}(F, M) \to \mathrm{Hom}_{\mathcal{D}}(E, M)$ 〃 $)$

これにより, $E, G \in (\mathcal{D}^{\leq 0})^{\perp}$ ならば,

$E \to F \to G \xrightarrow{+1}$ dist. tri. をみたす $F \in \mathcal{D}$ についても $F \in (\mathcal{D}^{\leq 0})^{\perp}$.

$F \in \mathcal{D}^{\geq 1}$ を任意にとる. $F \in (\mathcal{D}^{\leq 0})^{\perp}$ を示す.

(i) F の長さが 1 のとき.

$$\begin{array}{c} 0 \to F \\ \ B_1 \end{array}$$ より. $F \simeq B_1 \in \mathcal{A}[l_1]$ s.t. $0 > l_1$

$\forall E \in \mathcal{D}^{\leq 0}$, $\mathrm{Hom}_{\mathcal{D}}(E, F) = 0$ を示す.

(ア) E の長さも 1 のとき.

$$\begin{array}{c} 0 \to E \\ \ A_1 \end{array}$$ より. $E \simeq A_1 \in \mathcal{A}[k_1]$ s.t. $k_1 \geq 0$ $(> l_1)$ なので.

1つ目の仮定より. $\mathrm{Hom}_{\mathcal{D}}(E, F) = 0$.

(イ) Eの長さ $m>1$ のとき

mまでOKとして $m+1$ のとき.

$$E_m \to E \to A_{m+1} \xrightarrow{+1} \quad \text{dist. tri.}$$

s.t. $A_{m+1} \in \mathcal{A}[k_{m+1}]$, $k_m > k_{m+1} \geq 0$.

今,

$$\mathrm{Hom}_{\mathcal{D}}(A_{m+1}, F) \to \mathrm{Hom}_{\mathcal{D}}(E, F) \to \mathrm{Hom}_{\mathcal{D}}(E_m, F)$$

が完全で. (ア)より $\mathrm{Hom}_{\mathcal{D}}(A_{m+1}, F)=0$, 仮定より $\mathrm{Hom}_{\mathcal{D}}(E_m, F)=0$.

$\therefore$ $\mathrm{Hom}_{\mathcal{D}}(E, F)=0$. $\therefore$ (ア)(イ)より $F \in (\mathcal{D}^{\leq 0})^{\perp}$.

(ii) Fの長さ $n>0$ のとき.

nまでOKと仮定して $n+1$ のとき.

$$F_n \to F \to B_{n+1} \xrightarrow{+1} \quad \text{dist. tri.}$$

s.t. $B_{n+1} \in \mathcal{A}[l_{n+1}]$, $0 > l_{n+1}$

仮定より $F_n \in (\mathcal{D}^{\leq 0})^{\perp}$. (i)より $B_{n+1} \in (\mathcal{D}^{\leq 0})^{\perp}$

$\therefore$ $F \in (\mathcal{D}^{\leq 0})^{\perp}$

以上により $\mathcal{D}^{\geq 1} \subset (\mathcal{D}^{\leq 0})^{\perp}$.

Rem

より強く $\mathcal{D}^{\geq 1} = (\mathcal{D}^{\leq 0})^{\perp}$

∵ $(\mathcal{D}^{\leq 0})^{\perp} \subset \mathcal{D}^{\geq 1}$ を示す.

$\forall F \in (\mathcal{D}^{\leq 0})^{\perp}$ は

$\forall E \in \mathcal{D}^{\leq 0}$, $\mathrm{Hom}_{\mathcal{D}}(E, F) = 0$ をみたす

もし. $F \notin \mathcal{D}^{\geq 1}$ なる.

$\exists\ k_1 > k_2 > \cdots > k_i \geqslant 0 > k_{i+1} > \cdots > k_n$

s.t.

$$
\begin{tikzcd}
F_0 = 0 \arrow[r] & F_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & F_i \arrow[r] \arrow[dl] & F_{i+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & F_n = F \arrow[dl] \\
B_1 \arrow[u] & & B_i \arrow[ur] & B_{i+1} \arrow[u] & & B_n \arrow[u] & \\
A[k_1] \arrow[u, phantom, "\cong" sloped] & & A[k_i] \arrow[u, phantom, "\cong" sloped] & A[k_{i+1}] \arrow[u, phantom, "\cong" sloped] & & A[k_n] \arrow[u, phantom, "\cong" sloped] &
\end{tikzcd}
$$

このとき. $E \in \mathcal{D}^{\leq 0}$ として F_i をとると. $\left(F_i \xrightarrow{\iota} F \to \mathrm{Cone}(\iota) \xrightarrow{+1} \right)$

$$
\begin{array}{ccccc}
\mathrm{Hom}(F_i, F_i) & \to & \mathrm{Hom}(F_i, F) & \to & \mathrm{Hom}(F_i, \mathrm{Cone}(\iota)) \quad \text{exact} \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\
\mathrm{id} & \longmapsto & \iota & &
\end{array}
$$

$\mathrm{Hom}(F_i, F) \neq 0$ となり矛盾. $\quad \therefore\ F \in \mathcal{D}^{\geq 1}$

$\square$

Claim 3

$^\forall E \in \mathcal{D}$, $\exists\ F \to E \to G \xrightarrow{+1}$ in $\mathcal{D}$

s.t. $F \in \mathcal{D}^{\leq 0}$, $G \in \mathcal{D}^{\geq 1}$

∵ 仮定より $\exists!\ k_1 > \cdots > k_n \ (\in \mathbb{Z})$

$$\begin{tikzcd}[column sep=small]
\exists!\ 0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_i \arrow[rr] & & E_{i+1} \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul] \arrow[u, phantom, "\triangledown"] & & A_i \arrow[ul] & & A_{i+1} \arrow[ul] & & & & A_n \arrow[ul] &
\end{tikzcd}$$

s.t. $A_1 \in \mathcal{A}[k_1]$, $A_i \in \mathcal{A}[k_i]$, $A_{i+1} \in \mathcal{A}[k_{i+1}]$, $A_n \in \mathcal{A}[k_n]$

場合分け

(i) $k_1 > k_2 > \cdots > k_i \geq 0 > k_{i+1} > \cdots > k_n$

(ii) $0 \geq k_1 > \cdots > k_n$

(iii) $k_1 > k_2 > \cdots > k_n \geq 0$

(i) のとき. $F := E_i$ とおくと.

$$F \xrightarrow{\iota} E \to \mathrm{Cone}(\iota) \xrightarrow{+1}$$

s.t. $F \in \mathcal{D}^{\leq 0}$, $\mathrm{Cone}(\iota) =: G \in (\mathcal{D}^{\leq 0})^{\perp} = \mathcal{D}^{\geq 1}$

Claim2 と同じ手法

(ii) のとき. $E \simeq G$, $F = 0$.

(iii) のとき. $E \simeq F$, $G = 0$.

Claim 4

$(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ は 有界な t-structure.

(∵)

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} (\mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq j})$$ を示す.

$[\supset]$ は 明.

$[\subset]$ について. $\forall E \in \mathcal{D}$, $\exists K_1, K_n \in \mathbb{Z}$

s.t. $E \in \mathcal{D}^{\leq -K_n} \cap \mathcal{D}^{\geq -K_1}$

ここで. $K_1, K_n \in \mathbb{Z}$ は. 条件により E に対して存在する整数列

$K_1 > \cdots > K_n$ の中で 最大のものと 最小のもの

であり. Claim 2 の Rem から,

$$\mathcal{D}^{\leq -K_n} = \left\{ E \in \mathcal{D} \;\middle|\; \begin{array}{c} \exists!\, k_1 > k_2 > \cdots > k_m \geq K_n \\ \exists!\, 0 = E_0 \to E_1 \to \cdots \to E_m = E \\ \quad A_1 \qquad\qquad A_m \\ \text{s.t. } \forall j,\ A_j \in \mathcal{A}[k_j]. \end{array} \right\}$$

$$\mathcal{D}^{\geq -K_1} = \left\{ E \in \mathcal{D} \;\middle|\; \begin{array}{c} K_1 \geq^{\exists!} k_1 > \cdots > k_m \\ \exists!\, 0 = E_0 \to E_1 \to \cdots \to E_m = E \\ \quad A_1 \qquad\qquad A_m \\ \text{s.t. } \forall j,\ A_j \in \mathcal{A}[k_j]. \end{array} \right\}$$

□

[Proof of 7.7 (ii)]

$$\begin{array}{cccc}\Psi : & K(\mathcal{D}) & \longrightarrow & K(\mathcal{A}) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & [E] & \longmapsto & \sum_{i\in\mathbb{Z}} (-1)^i [\mathcal{H}_{\mathcal{A}}^i(E)]\end{array}$$

と定める.

① Ψ は well-defined.

$[E]=[F] \in K(\mathcal{D})$ なら $\forall i$, $\mathcal{H}_{\mathcal{A}}^i(E)=\mathcal{H}_{\mathcal{A}}^i(F)$.

$\therefore$ $\Psi([E])=\Psi([F])$.

② Ψ は 群準同型.

$\mathcal{D}$ で dist. tri.

$$E \to F \to G \xrightarrow{+1} E[1]$$ があれば.

$\mathcal{H}_{\mathcal{A}}^i(-)$ が cohomological functor なので.

$$\begin{aligned} \cdots \to \mathcal{H}_{\mathcal{A}}^i(E) &\to \mathcal{H}_{\mathcal{A}}^i(F) \to \mathcal{H}_{\mathcal{A}}^i(G) \\ \to \mathcal{H}_{\mathcal{A}}^{i+1}(E) &\to \mathcal{H}_{\mathcal{A}}^{i+1}(F) \to \mathcal{H}_{\mathcal{A}}^{i+1}(G) \to \cdots \end{aligned}$$

は $\mathcal{A}$ での exact 列.

よって、

$$
\begin{aligned}
\Psi([F]) &= \sum_{i\in\mathbb{Z}} (-1)^i [\mathcal{H}^i_{\mathcal{A}}(F)] \\
&= \sum (-1)^i \{[\mathcal{H}^i_{A}(E)] + [\mathcal{H}^i_{\mathcal{A}}(G)]\} \\
&= \Psi([E]) + \Psi([G]).
\end{aligned}
$$

③ Ψ は全単射で、逆写像も群準同型.

$$
\begin{aligned}
\varphi \quad : \quad K(\mathcal{A}) &\longrightarrow K(\mathcal{D}) \\
[E] &\longmapsto [E]
\end{aligned}
$$

とおく。 φ は明らかに群準同型.

$\Psi \circ \varphi = \mathrm{id}$, $\varphi \circ \Psi = \mathrm{id}$ を示すためには。 $K(\mathcal{A})$ で

$[E] = \sum_{i\in\mathbb{Z}} (-1)^i [\mathcal{H}^i_{\mathcal{A}}(E)]$ を示せばよい.

(i) $E \in \mathcal{D}^{\leq m} \cap \mathcal{D}^{\geq m}$ のとき.

$$\mathcal{H}^m_A(E) = (\tau^{\geq m}\overset{\overset{E}{\wr\!\mid}}{\tau^{\leq m}})(E)[m]$$

このとき、 $E = \mathcal{H}^m_{\mathcal{A}}(E)[-m]$ なので、

$[E] = [\mathcal{H}^m_{A}(E)[-m]] = (-1)^m [\mathcal{H}^m_{\mathcal{A}}(E)]$ となり ok.

（一般に、 $X \to X \to 0 \to X[1]$ が dist. tri なので、

$[X] = -[X[1]]$ ）

(ii) 一般の $E \in \mathcal{D}$ について

$\mathcal{A}$ は bounded t-structure なので $\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} (\mathcal{D}^{\leq i} \cap \mathcal{D}^{\geq j})$

$m \geq n$ として $E \in \mathcal{D}^{\leq m} \cap \mathcal{D}^{\geq n}$ とする.

n を固定して m に関する帰納法で示す.

$n = m$ のときは (i) と同じなので, $m > n$ とする.

また, $m-1$ まで主張が正しいとする.

$$\tau_{\leq m-1}(E) \to E \to \tau_{\geq m}(E) \xrightarrow{+1}$$

$$(\tau_{\leq m-1}(E) \in \mathcal{D}^{\leq m-1} \cap \mathcal{D}^{\geq n}, \quad \tau_{\geq m}(E) = \mathcal{D}^{\leq m} \cap \mathcal{D}^{\geq m})$$

なる dist. tri. があり. (cf. HTT(D-加群) Prop 8.1.5(ii)) $\tau_{\leq m-1}(E)$ については仮定より

$$[\tau_{\leq m-1}(E)] = \sum_{i=n}^{m-1} (-1)^i [\mathcal{H}^i_{\mathcal{A}}(\tau_{\leq m-1}(E))]$$
$$= \sum_{i=n}^{m-1} (-1)^i [\mathcal{H}^i_{\mathcal{A}}(E)]$$

$\tau_{\geq m}(E)$ についても, (i) より

$$[\tau_{\geq m}(E)] = (-1)^m [\mathcal{H}^m_{\mathcal{A}}(E)]$$

$$\therefore \quad [E] = [\tau_{\leq m-1}(E)] + [\tau_{\geq m}(E)]$$
$$= \sum_{i=n}^{m-1} (-1)^i [\mathcal{H}^i_{\mathcal{A}}(E)] + (-1)^m [\mathcal{H}^i_{\mathcal{A}}(E)]$$
$$= \sum_{i=n}^{m} (-1)^i [\mathcal{H}^i_{\mathcal{A}}(E)]$$

□