Quiver の表現論

$Q = (Q_0, Q_1)$; quiver.

Q_0 : vertices. Q_1 : arrows

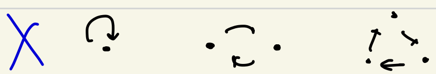
例 $\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet}$ A_2-quiver, $Q_0 = \{1, 2\}$, $Q_1 = \{a\}$.

$\overset{1}{\bullet} \underset{a_2}{\overset{a_1}{\rightrightarrows}} \overset{2}{\bullet}$ $\hat{A}_1$-quiver (Kronecker quiver とも呼ばれる)

$Q_0 = \{1, 2\}$, $Q_1 = \{a_1, a_2\}$

仮定 (表現の圏を簡単にするため)

・Q はループやサイクルを持たない.



・$|Q_0| < \infty$, $|Q_1| < \infty$.

Def

Q の体 $\Bbbk$ 上の有限次元表現とは,

・各 vertex $i \in Q_0$ に f.d. $\Bbbk$-vec. sp. V_i

・各 arrow $i \xrightarrow{a} j \in Q_1$ に $\Bbbk$-linear map $V_i \to V_j$

を対応させたもの.

例

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "a"] & \overset{2}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \overset{V_1}{\bullet} \arrow[r, "f_a"] & \overset{V_2}{\bullet} \end{tikzcd}$$

$\Bbbk = \mathbb{F}_q$ とする.

$$S_1 = \begin{tikzcd} \underset{1}{\overset{\mathbb{F}_q}{\bullet}} \arrow[r, "0"] & \underset{2}{\overset{\{0\}}{\bullet}} \end{tikzcd}$$

$$S_2 = \begin{tikzcd} \underset{1}{\overset{\{0\}}{\bullet}} \arrow[r, "0"] & \underset{2}{\overset{\mathbb{F}_q}{\bullet}} \end{tikzcd}$$

とすると. 任意の A_2-quiver の表現は. S_1, S_2 の有限回の extension

として表せる.

i.e., $\quad K(\mathrm{rep}_{\mathbb{F}_q}(\overset{1}{\bullet} \to \overset{2}{\bullet})) \cong \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$.

記号

$\mathrm{rep}_{\Bbbk}(Q)$ = category of f.d. rep. of $Q / \Bbbk$.

射について:

$(V_i, f_a), (W_i, g_a) \in \mathrm{rep}_{\Bbbk}(Q)$

$\varphi : (V_i, f_a) \to (W_i, g_a)$ は.

$\varphi = (\varphi_i)_{i \in Q_0}$ s.t.
$$\begin{tikzcd} V_i \arrow[r, "f_a"] \arrow[d, "\varphi_i"'] \arrow[dr, phantom, "\circlearrowleft"] & V_j \arrow[d, "\varphi_j"] \\ W_i \arrow[r, "g_a"'] & W_j \end{tikzcd}$$

A_2-quiver の indecomposable

$$T = \underset{1}{\overset{\mathbb{F}_q}{\bullet}} \xrightarrow{1} \underset{2}{\overset{\mathbb{F}_q}{\bullet}}, \qquad \begin{tikzcd} S_2 = & \overset{\{0\}}{\bullet} \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, phantom, "\circlearrowleft"] & \overset{\mathbb{F}_q}{\bullet} \arrow[d, "1"] \\ T = & \overset{\mathbb{F}_q}{\bullet} \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\circlearrowleft"] & \overset{\mathbb{F}_q}{\bullet} \arrow[d, "0"] \\ S_1 = & \overset{\mathbb{F}_q}{\bullet} \arrow[r, "0"] & \overset{\{0\}}{\bullet} \end{tikzcd}$$

$0 \to S_2 \to T \to S_1 \to 0$ という short exact sq. がある。

Thm

A_2-quiver の indecomposable object は S_1, S_2, T のみ,
他の object は この 3つの 有限個 の 直和 として表せる.

$$K(\mathrm{rep}_{\mathbb{F}_q}(A_2)) \cong \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$$

$$\alpha_1 := [S_1], \quad \alpha_2 := [S_2], \quad \alpha_1 + \alpha_2 = [T]$$

とおくと. $\{\text{indecomposable}\} \overset{1:1}{\longleftrightarrow} \{A_2\text{-root 系 の positive root}\}$.

一般に, ADE quiver でも正しい (Gabriel の定理)

ADE の外 (affine や indefinite) も正しい (Kac の定理)

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Quiver の表現の一般論

$Q = (Q_0, Q_1)$; Quiver

頂点　矢印

仮定

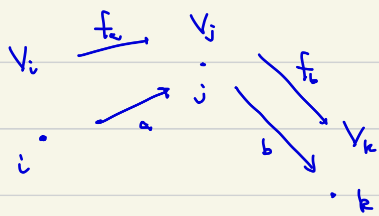
$|Q_0| < \infty,\ |Q_1| < \infty$, acyclic

k : 体

$\mathrm{rep}\, Q/k$ = category of finite dim'l rep. of Q $/k$.

Object

$$\begin{cases} (V_i)_{i \in Q_0}, & V_i : \text{finite dim'l } k\text{-vec. sp.} \\ (f_a)_{a \in Q_1}, & V_i \xrightarrow{f_a} V_j \text{ for } i \xrightarrow{a} j \quad (\text{linear}) \end{cases}$$



$$\left(V_i \xrightarrow{f_a} V_j \xrightarrow{f_b} V_k \right)$$

<u>morphism</u>

$$(\varphi_i)_{i\in Q_0} : \begin{cases} (V_i)_{i\in Q_0} \\ (f_a)_{a\in Q_1} \end{cases} \longrightarrow \begin{cases} (W_j)_{j\in Q_0} \\ (g_b)_{b\in Q_1} \end{cases}$$

$$\begin{tikzcd} V_i \arrow[r, "f_a"] \arrow[d, "\varphi_i"'] \arrow[dr, phantom, "\circlearrowright"] & V_j \arrow[r, "f_b"] \arrow[d, "\varphi_j"] \arrow[dr, phantom, "\circlearrowright"] & V_k \arrow[d, "\varphi_k"] \\ W_i \arrow[r, "g_a"'] & W_j \arrow[r, "g_b"'] & W_k \end{tikzcd}$$

rep Q/k の構造と $K(\text{rep } Q/k)$

(もし Q が acyclic でないときは、
rep$^{\text{nil}}$; nilpotent representations of Q を考えると同様.)

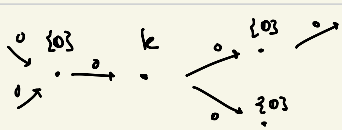
以下では Q : acyclic を仮定しておく.

Jordan-Hölder filtration

$S_i \in \text{rep } Q/k$ を $((V_j)_{j\in Q_0}, (f_a)_{a\in Q_1})$ s.t.

$$V_j = \begin{cases} k & j = i \\ 0 & j \neq i, \end{cases} \qquad \forall a \in Q_1,\ f_a = 0$$

として定める.



i.e., 頂点 i にのみ 1次元 k がのっていて,

全ての arrow a の linear map $f_a = 0$ となるのが S_i.

性質

- $\dim_k S_i = 1$
- simple, つまり, proper sub rep. を持たない.

Prop (Jordan Hölder filtration)

$\forall M \in \mathrm{rep}\, Q/k$ に対し,

$\exists$ filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_d = M. \quad (d = \dim_k M)$$

s.t. $M_i / M_{i-1} \in \{ S_j \mid j \in Q_0 \}$

Rem

$$\dim_k M = \sum_{i \in Q_0} \dim_k V_i$$

$\{ \mathrm{rep}\, Q/k$ の simple rep の同型類$\} = \{ S_j \mid j \in Q_0 \}$

(indecomposable なものはいっぱいある…?)

Rem

単純 = simple = Q_0 でパラメトライズされる.

直既約 = indecomposable = Q のルート系と関係

($M = M_1 \oplus M_2$ と直和分解できない)

$\begin{cases} Q = ADE \Rightarrow \text{正ルートと } 1:1 & \text{Gabriel} \\ Q \neq ADE \Rightarrow \text{dim vector が正ルート上にある} & \text{Kac} \end{cases}$

(例: $\overset{k}{\bullet} \overset{z}{\underset{w}{\rightrightarrows}} \overset{k}{\bullet} \leftrightarrow \mathcal{O}_x, \ x \in \mathbb{P}^1_k$

$[z, w] \in \mathbb{P}^1_k$)

[ref: Kac ; Infinite root systems, rep. of graph. and invariant theory]
(c1980. Invention)

Cor

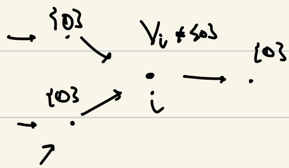
$$\begin{array}{ccccc} K(\operatorname{rep} Q/k) & \cong & \bigoplus_{i \in Q_0} \mathbb{Z}[S_i] & \cong & \mathbb{Z}^{Q_0} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [M] = [(V_i)_{i \in Q_0}] & \mapsto & \sum_{i \in Q_0} \dim_k V_i [S_i] & \longmapsto & \underline{\dim} M \end{array}$$

(ただし. $\underline{\dim}\, M := (\dim_k V_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$)

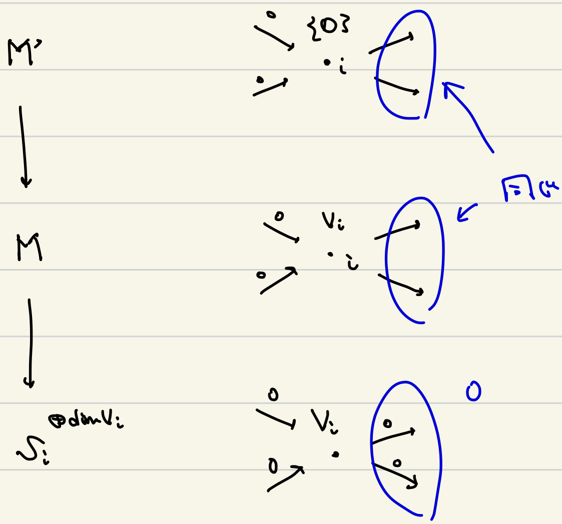
Prp (JH) の Proof

$i \in Q_0$ で $\begin{cases} \cdot\ V_i \neq 0 \\ \cdot\ j \text{ から } i \text{ に path がある} \Rightarrow V_j = 0 \end{cases}$

となるものをとる.



$0 \to M' \to M \to S_i^{\oplus \dim V_i} \to 0$ (exact) がい



$\rightsquigarrow$ $0 \subset M' \subset M$, $M/M' \cong S_i^{\oplus \dim V_i}$, $\dim M' < \dim M$

$\rightsquigarrow$ induction

□

rep Q/k の Ext

- $\mathrm{Hom}(S_i, S_j) = \begin{cases} k & (i = j) \\ 0 & (i \neq j) \end{cases}$

記号

$q_{ij} = \#\{ a \in Q_1 \mid i \xrightarrow{a} j \}$ （i から j への矢の数）.

Prop

$\mathrm{Ext}^1(S_i, S_j) \cong k^{q_{ij}}$

(ref : Elements of the Rep. theory of ass. alg. 1, § II. 2. Lem 2.12)
?

Thm

$\forall E, F \in \mathrm{rep}\, Q/k$, $\mathrm{Ext}^i(E, F) = 0$ for $\forall i \geq 2$

(ref : 同じ本 § VII. 1, Thm 1.7).

Def

Abel 圏 $\mathcal{A}$ が hereditary
$\overset{\mathrm{def}}{\iff}$ $\forall E, F \in \mathcal{A}$, $\mathrm{Ext}^i(E, F) = 0$ $i \geq 2$.

Cor

rep (Q/k) は hereditary

⑩ Hereditary で嬉しいこと.

$D^b(\mathcal{A})$ の object の分類が簡単

Thm

$E \in D^b(\mathcal{A})$ が indecomposable
$\Rightarrow \exists E' \in \mathcal{A}$; indecomposable in $\mathcal{A}$
s.t. $E = E'[n]$

(ref : Happel の三角圏の本)

$\mathcal{A} = \mathrm{rep}\, Q/\mathbb{F}_q$ の Hall alg. $H(\mathcal{A})$ の積分

まず,
- Torus $\mathbb{T}_+$
- twisted torus $\mathbb{T}_-$
- quantum torus $\mathbb{T}_q$ (← 実体はない)

の座標環を導入する.

設定

- $\Gamma = \mathbb{Z}^{\oplus n}$
- $\langle -, - \rangle : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$; bilinear anti-symmetric

$$(\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle$$

が与えられているとする.

例

$\mathcal{D}$: CY3 triangulated cat.

$\Gamma = K(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$ (仮定記)

$\forall E, F \in \mathcal{D}$, $\sum_i \dim \mathrm{Hom}(E, F[i]) < \infty$ (仮定).

$$\langle [E], [F] \rangle = \chi(E,F) = \sum_i (-1)^i \dim \mathrm{Hom}(E, F[i])$$

Rem

$\mathcal{D}$ が $\underline{\text{CY } N}$

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall E, F \in \mathcal{D}$, $\mathrm{Hom}(E,F) \xrightarrow[\sim]{\text{natural}} \mathrm{Hom}(F, E[N])^*$.

記号

$$\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^n$$

$$z \longmapsto (z_1, z_2, \cdots, z_n)$$

$\underset{(\alpha_1, \cdots, \alpha_n)}{\overset{\parallel}{\alpha}} \in \Gamma \cong \mathbb{Z}^n$, $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ $(= z(\alpha))$.

$e_i = (0, 0, \cdots, 0, \underset{\text{ⓘ}}{1}, 0, \cdots, 0)$, $z_i = z(e_i)$

Def

$\mathbb{C}[\mathbb{T}_+]$, $\mathbb{C}[\mathbb{T}_-]$, $\mathbb{C}[\mathbb{T}_q]$ を次で定める.

as vector spaces

$$\mathbb{C}[\mathbb{T}_+] = \mathbb{C}[\mathbb{T}_-] = \bigoplus_{\alpha\in\Gamma} \mathbb{C} y^\alpha$$

$$\mathbb{C}[\mathbb{T}_q] = \bigoplus_{\alpha\in\Gamma} \mathbb{C}[q^{\pm\frac{1}{2}}] y^\alpha$$

ring structures

可換

- $\mathbb{C}[\mathbb{T}_+] = \mathbb{C}[y_1^{\pm1}, y_2^{\pm1}, \dots, y_n^{\pm1}]$ s.t. $y^\alpha \cdot y^\beta := y^{\alpha+\beta}$
- $\mathbb{C}[\mathbb{T}_-] = \mathbb{C}[y_1^{\pm1}, y_2^{\pm1}, \dots, y_n^{\pm1}]$ s.t. $y^\alpha \cdot y^\beta := (-1)^{\langle\alpha,\beta\rangle} y^{\alpha+\beta}$
- $\mathbb{C}[\mathbb{T}_q]$ $\quad y^\alpha \cdot y^\beta := q^{\frac{1}{2}\langle\alpha,\beta\rangle} y^{\alpha+\beta}$

  $(\Rightarrow y^\alpha \cdot y^\beta = q^{\langle\alpha,\beta\rangle} y^\beta \cdot y^\alpha)$ 一般に非可換.

Poisson structures

- $\{-,-\} : \mathbb{C}[\mathbb{T}_\pm] \otimes \mathbb{C}[\mathbb{T}_\pm] \longrightarrow \mathbb{C}[\mathbb{T}_\pm]$ を,

  $\{y^\alpha, y^\beta\} := \langle\alpha,\beta\rangle y^{\alpha+\beta}$ とおく.

  $\mathbb{C}[\mathbb{T}_\pm]$ は可換 Poisson ring → $\mathbb{C}[T_-]$ のときは

  $\{y^\alpha, y^\beta\} = (-1)^{\langle\alpha,\beta\rangle}\langle\alpha,\beta\rangle y^{\alpha+\beta}$

- $\mathbb{C}[\mathbb{T}_q]$ は ($q-1$ で Localize して) ← localization 必要なし. (12/5)

$$\{y^\alpha, y^\beta\} := \frac{y^\alpha\cdot y^\beta - y^\beta\cdot y^\alpha}{q-1}$$

$$= \frac{q^{\frac{1}{2}\langle\alpha,\beta\rangle} - q^{-\frac{1}{2}\langle\alpha,\beta\rangle}}{q^{\frac{1}{2}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} y^{\alpha+\beta}$$ とおく.

(キモチ)

motivic DT

$$\begin{tikzcd}
 & \mathbb{C}[\mathbb{T}_q] \arrow[dl, "{q^{\frac{1}{2}} \to 1 \text{ になる},\ q \to 1 \text{ の limit}}"'] \arrow[dr, "{q^{\frac{1}{2}} \to -1 \text{ となる},\ q \to 1 \text{ の limit}}"] & \\
\mathbb{C}[\mathbb{T}_+] & & \mathbb{C}[\mathbb{T}_-]
\end{tikzcd}$$

Euler 数　　　　　　　　DT 不変量

Rem

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^\times) = \mathrm{Spec}(\mathbb{C}[\mathbb{T}_+])$$

$$\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^\times) = \{ g : \Gamma \to \mathbb{C}^\times \mid g(\alpha+\beta) = g(\alpha) g(\beta) \}$$

$$\mathbb{T}_- = \{ g : \Gamma \to \mathbb{C}^\times \mid g(\alpha+\beta) = (-1)^{\langle \alpha, \beta \rangle} g(\alpha) g(\beta) \} = \mathrm{Spec}(\mathbb{C}[\mathbb{T}_-])$$

$$\mathbb{T}_q \text{ “=” } \mathrm{Spec}(\underbrace{\mathbb{C}[\mathbb{T}_q]}_{})$$

非可換なので空間としての実体はない.

ref: torus, twisted torus などの setting は Bridgeland の BPS structure からとってきた. (Riemann-Hilbert problems from DT theory の最初の方).

quantum torus は. (A quantized RH problems from DT theory).

12/5

復習

(BPS structure の一部)

- $\Gamma = \mathbb{Z}^n$
- $\langle -,- \rangle : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$; bilinear, anti-symmetric.

に対し、

- $\mathbb{C}[T_+] = \mathbb{C}[T_-] = \bigoplus_{\alpha \in \Gamma} \mathbb{C} y^\alpha$.
- $\mathbb{C}[T_q] = \bigoplus_{\alpha \in \Gamma} \mathbb{C}[q^{\pm \frac{1}{2}}] y^\alpha$.

($\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$ に対し、 $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$
ただし、 $y_i = y^{e_i}$, $e_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0) \in \mathbb{Z}^n$)

↑ この解釈は $\mathbb{C}[T_+]$ のときだけとする.

これに、ring structure を

<u>$\mathbb{C}[T_+]$</u>

$$y^\alpha \cdot y^\beta := y^{\alpha+\beta}$$

<u>$\mathbb{C}[T_-]$</u>

$$y^\alpha \cdot y^\beta := (-1)^{\langle \alpha, \beta \rangle} y^{\alpha+\beta}$$

可換環

<u>$\mathbb{C}[T_q]$</u>

$$y^\alpha \cdot y^\beta := q^{\frac{1}{2}\langle \alpha, \beta \rangle} y^{\alpha+\beta}$$

一般には非可換環.

で定める.

Poisson structure on $\mathbb{C}[T_\varepsilon]$, $(\varepsilon = +, -, q)$ (前回から訂正).

$\mathbb{C}[T_+]$

$$\{ y^\alpha, y^\beta \} = \langle \alpha, \beta \rangle y^{\alpha+\beta}$$

$\mathbb{C}[T_-]$

$$\{ y^\alpha, y^\beta \} = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle y^{\alpha+\beta}$$

$\mathbb{C}[T_q]$

($q-1$ で localize する必要なし.)

$$\{ y^\alpha, y^\beta \} = \frac{y^\alpha \cdot y^\beta - y^\beta \cdot y^\alpha}{q-1}$$

$$= \frac{q^{\frac{1}{2}\langle \alpha, \beta \rangle} - q^{-\frac{1}{2}\langle \alpha, \beta \rangle}}{q^{\frac{1}{2}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} y^{\alpha+\beta}$$

$$= q^{-\frac{1}{2}} \cdot \underbrace{\frac{q^{\frac{1}{2}\langle \alpha, \beta \rangle} - q^{-\frac{1}{2}\langle \alpha, \beta \rangle}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}}_{\in \mathbb{C}[q^{\pm\frac{1}{2}}]} y^{\alpha+\beta}$$

$q^{\frac{1}{2}} \to 1$: $\quad \langle \alpha, \beta \rangle y^{\alpha+\beta}$

$q^{\frac{1}{2}} \to -1$: $\quad (-1)^{\langle \alpha+\beta \rangle} \langle \alpha, \beta \rangle y^{\alpha+\beta}$

Rem

$$\frac{q^{\langle \alpha, \beta \rangle} - 1}{q-1} \xrightarrow{q^{\frac{1}{2}} \to \pm 1} \langle \alpha, \beta \rangle$$

つまり、

Poisson ring として、

motivic DT

q-DT

$$\begin{tikzcd}
 & \mathbb{C}[T_q] \arrow[dl, "q^{\frac{1}{2}} \to +1"'] \arrow[dr, "q^{\frac{1}{2}} \to -1"] & \\
\mathbb{C}[T_+] & & \mathbb{C}[T_-]
\end{tikzcd}$$

Euler数 　　　　　　　　　　DT

が得られる。

<u>$\mathcal{A} = \mathrm{Rep}\, Q / \mathbb{F}_q$ の Hall 代数の積分 (Reineke)</u>

$Q = (Q_0, Q_1)$; acyclic quiver.

$$\begin{array}{ccc}
K(\mathrm{rep}\, Q/\mathbb{F}_q) & \cong & \mathbb{Z}^{Q_0} \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
[V] & \longmapsto & \underline{\dim} V = (\dim V_i)_{i \in Q_0} \\
\| & & \\
[(V_i)_{i \in Q_0}] & &
\end{array}$$

$$[S_i] \longmapsto e_i = (0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$$

$$q_{ij} := \#\{ a \in Q_1 \mid i \xrightarrow{a} j \}$$
$$= (i \text{ から } j \text{ への矢印の本数})$$ とおく.

Exchange matrix B_Q を,

$$(B_Q)_{ij} = q_{ij} - q_{ji}$$

とし.

$$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}$$

を,

$$\langle e_i, e_j \rangle := (B_Q)_{ij} = q_{ij} - q_{ji}$$

と定める.

<u>Rem</u>

$$\begin{cases} \chi(S_i, S_j) = \delta_{ij} - q_{ij} \\ \dim \mathrm{Hom}(S_i, S_j) = \delta_{ij} \\ \dim \mathrm{Ext}^1(S_i, S_j) = q_{ij} \end{cases}$$

より,

$$\langle e_i, e_j \rangle = \chi(S_j, S_i) - \chi(S_i, S_j).$$

例

$$Q = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \qquad B_Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Q = \overset{1}{\bullet} \rightrightarrows \overset{2}{\bullet} \qquad B_Q = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

■

$$H(\mathcal{A}) = \{ f : \underset{\mathrm{ob}(\mathcal{A})/\sim}{\overset{\shortparallel}{M_{\mathcal{A}}}} \longrightarrow \mathbb{C} \mid \mathrm{Supp}(f) \text{ は有限} \}$$

Def

$$I_q : \overset{\mathrm{rep}\,Q/\mathbb{F}_q}{\overset{\shortparallel}{H(\mathcal{A})}} \longrightarrow \mathbb{C}[T_q]$$

$\mathbb{Z}^{Q_0}$
$\langle e_i, e_j \rangle = q_{ij} - q_{ji}$
から定まる quantum torus

を、

$$I_q(f) := \sum_{V \in M_{\mathcal{A}}} \frac{q^{\frac{1}{2}\chi(V,V)} f(V)}{|\mathrm{Aut}(V)|} y^{\underline{\dim} V}$$

と定める。 $\chi(V,V) = \dim \mathrm{Hom}(V,V) - \dim \mathrm{Ext}^1(V,V)$.

Prop

I_q は $\mathbb{C}$-alg. hom.

Lem

$$
S_i = \begin{tikzcd}[column sep=small]
0 \arrow[dr, "0"] & & 0 \\
\vdots & \overset{\mathbb{F}_q}{\underset{i}{\bullet}} \arrow[ur] \arrow[dr] & \vdots \\
0 \arrow[ur, "0"] & & 0
\end{tikzcd}
$$

に対し

$$\delta_{S_i^{\oplus n}} \in H(A)$$

が.

$$\delta_{S_i^{\oplus n}}(V) = \begin{cases} 1 & V \cong S_i^{\oplus n} \\ 0 & \text{otherwise} \end{cases}$$

で定まる. このとき.

$$I_q(\delta_{S_i^{\oplus n}}) = \frac{q^{\frac{1}{2}n}}{(q-1)(q^2-1)\cdots(q^n-1)} y_i^n$$

$(y_i := y^{e_i}, \quad y^{e_i}\cdot y^{e_i} = y^{2e_i}, \quad \langle e_i, e_i\rangle = 0).$

∵

$$I_q(\delta_{S_i^{\oplus n}}) = \sum_{V\in M_A} \frac{q^{\frac{1}{2}\chi(V,V)}\delta_{S_i^{\oplus n}}(V)}{|\mathrm{Aut}(V)|} y^{\underline{\dim} S_i^{\oplus n}}$$

$$= \frac{q^{\frac{1}{2}\chi\langle S_i^{\oplus n}, S_i^{\oplus n}\rangle}}{|\mathrm{Aut}(S_i^{\oplus n})|} y^{\underline{\dim} S_i^{\oplus n}}$$

ここで, $|\mathrm{Aut}(S_i^{\oplus n})| = |\mathrm{Aut}(\mathbb{F}_q^n)| = |GL_n(\mathbb{F}_q)|$

$$= (q-1)(q^2-1)\cdot\cdots\cdot(q^n-1)\, q^{\frac{1}{2}n(n-1)}$$

$\cdot\ \chi\langle S_i^{\oplus n}, S_i^{\oplus n}\rangle = \chi\langle ne_i, ne_i\rangle = n^2$

$$y^{\underline{\dim}\, S_i^{\oplus n}} = y^{ne_i} = y_i^n$$

これらから示される。 (Keller p.9 も参照). □

A_2-quiver の例 と quantum dilog identity

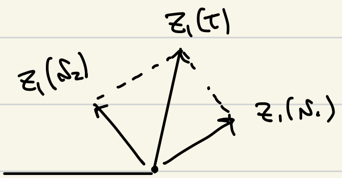
$Q = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ $\qquad B_Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$y_1 \cdot y_2 = q^{\frac{1}{2}} y^{e_1+e_2}$$
$$y_2 \cdot y_1 = q^{-\frac{1}{2}} y^{e_1+e_2}$$

より、 $y_1 \cdot y_2 = q \cdot y_2 \cdot y_1$ (非可換).

$\mathcal{A} = \mathrm{rep}\, Q / \mathbb{F}_q$ 上に 2つの stability を考える.

①



$T = \overset{\mathbb{F}_q}{\bullet} \xrightarrow{1} \overset{\mathbb{F}_q}{\bullet}$

$0 \to S_2 \to T \to S_1 \to 0$ (ex).

という central charge Z_1 を考える.

$\rightsquigarrow$ S_1, S_2 は stable. T は unstable.

($\because$ $S_2 \subset T$ に対し $\phi(S_2) > \phi(T)$)

HN identity

$$\delta_{\mathcal{A}} = \delta^{Z_1\text{-}ss}(\phi(S_2)) * \delta^{Z_1\text{-}ss}(\phi(S_1))$$

Stable objects は S_1, S_2

semi stable は $S_1^{\oplus n}$, $S_2^{\oplus m}$

$$\begin{cases} \delta^{Z_1\text{-}ss}(\phi(S_2)) = \delta_{\mathcal{A}_2} = \sum_{n=0}^{\infty} \delta_{S_2^{\oplus n}} \\ \delta^{Z_1\text{-}ss}(\phi(S_1)) = \delta_{\mathcal{A}_1} = \sum_{n=0}^{\infty} \delta_{S_1^{\oplus n}} \end{cases}$$

ここで. $\mathcal{A}_i = \langle S_i \rangle = \{ S_i^{\oplus n} \mid n \geq 0 \}$

S_i の生成する full sub category

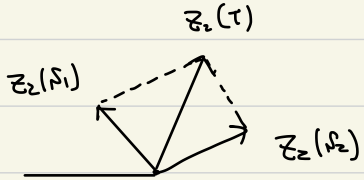
$$\delta_{\mathcal{A}_i}(V) = \begin{cases} 1 & V \in \mathcal{A}_i \\ 0 & \text{otherwise} \end{cases}$$

よって. 積分すると,

$$\begin{aligned} I_q(\delta_{\mathcal{A}}) &= I_q(\delta_{\mathcal{A}_2} * \delta_{\mathcal{A}_1}) \\ &= I_q(\delta_{\mathcal{A}_2}) \cdot I(\delta_{\mathcal{A}_1}) \\ &= \mathbb{E}_q(y_2) \cdot \mathbb{E}_q(y_1) \end{aligned}$$

ここで. $\mathbb{E}_q(y) = \sum_{n=0}^{\infty} \dfrac{q^{\frac{1}{2}n}}{(q-1)(q^2-1)\cdots(q^n-1)} y^n$

②



という central charge を考えると。

S_1, S_2, T は stable.

<u>HN identity</u>

$$\delta_{\mathcal{A}} = \delta^{Z_2\text{-}ss}(\phi(S_1)) * \delta^{Z_2\text{-}ss}(\phi(T)) * \delta^{Z_2\text{-}ss}(\phi(S_2))$$

$$= \delta_{\mathcal{A}_1} * \delta_{\mathcal{B}} * \delta_{\mathcal{A}_2}$$

ここで $\quad \mathcal{B} = \langle T \rangle = \{ T^{\oplus n} \mid n \geq 0 \}$ より

$$I_q(\delta_{\mathcal{A}}) = I_q(\delta_{\mathcal{A}_1}) I_q(\delta_{\mathcal{B}}) I_q(\delta_{\mathcal{A}_2})$$

$$= \mathbb{E}_q(y_1) \mathbb{E}_q(\delta_{\mathcal{B}}) \mathbb{E}_q(y_2)$$

$$y^{\underline{\dim} T} = y^{e_1 + e_2}$$

$$y_1 \cdot y_2 = q^{\frac{1}{2}} y^{e_1 + e_2} \quad \text{より} \quad y^{e_1 + e_2} = q^{-\frac{1}{2}} y_1 y_2$$

$$\therefore \quad I_q(\delta_{\mathcal{B}}) = \mathbb{E}_q(q^{-\frac{1}{2}} y_1 y_2)$$

① と ② で 2通りの $I_q(\delta_A)$ の表示を得た.

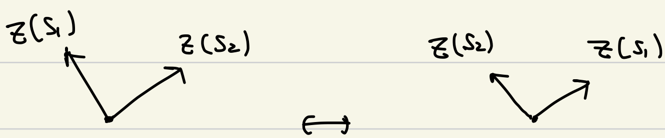
$$\mathbb{E}_q(y_2)\,\mathbb{E}_q(y_1) = \mathbb{E}_q(y_1)\,\mathbb{E}_q(q^{-\frac{1}{2}}y_1y_2)\,\mathbb{E}_q(y_2).$$

$$\boxed{y_1y_2 = q\cdot y_2y_1}$$

quantum dilog identity

$$\delta_A = (Z_1 \text{ の HN identity}) = (Z_2 \text{ の HN identity})$$

wall crossing



S_1 と S_2 の phase の順の wall crossing

で T が stable から unstable に切り替わる.
(unstable) (stable)

classical dilog

$\sim \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ $\qquad \dfrac{1}{n^2}$ は DT inv. と思える.

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n}}{(q-1)(q^2-1)\cdots(q^n-1)}\, y^n$$

motivic DT と関係.

Bridgeland の reference の p.17

$$I : H_{\mathrm{motivic}}(\mathrm{rep}\,Q/\mathbb{C}) \longrightarrow \mathbb{C}[T_q]$$

これは、 $q^{\frac{1}{2}} = t$,

$$\chi_t(X) = \sum_{n \geq 0} t^n \dim H^i(X, \mathbb{C})$$

そして、 $I([S \to M_\alpha]) = \chi_t(S) \cdot x^\alpha$

$\mathbb{P}^1 \rightsquigarrow \# \mathbb{P}^1(\mathbb{F}_q) = q + 1$.

$\chi_t(\mathbb{P}^1(\mathbb{C})) = t^2 + 1$ （だいたい同じ）

$[\mathbb{P}^1(\mathbb{C})] = [\mathbb{C}] + [pt]$

$= q + 1$ $([\mathbb{C}] = [\mathbb{L}] = q$ として$)$.