§1 Moduli Space of Vector Bundles

§2 Hitchin System and Spectral Curves

§3 Non Abelian Hodge Theory

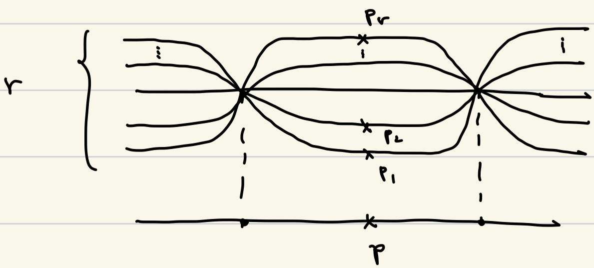
/ $\mathbb{C}$

§1

Σ : Compact non-singular algebraic curve of genus g
(Riemann Surface)

To motivate the introduction of moduli space of vector bundles, let us try to describe vector bundles on Σ come from the direct image of a line bdl.

$\pi : C \to \Sigma$: r-sheeted branched cover.



$$\begin{array}{ccc} C & \swarrow & L \\ \pi \downarrow & & \\ \Sigma & \swarrow & \pi_* L \end{array}$$

For $L \in Pic(C)$ (i.e., line bundle)

$E := \pi_* L$ is a rank r vec. bdl. on Σ

$\left(E_p \cong \bigoplus_{i=1}^{r} L_{p_i} \quad \text{for } p \notin \{\text{Ramification locus}\} \right.$

$\left. \pi^{-1}(p) = \{p_1, \cdots, p_r\} \right)$

[Jumping Phenomena]

Ex]

$\pi : C \longrightarrow \mathbb{P}^1$; 2-sheeted branched cover, $g(C) > 0$

If $\chi(L) = 0$ (i.e., $\deg(L) = g-1$ by Riemann-Roch)
we get $\deg(\pi_* L) = -2$ by R.R.

Let $l := h^0(C, L) = h^0(\mathbb{P}^1, \pi_* L)$. This implies
$\pi_* L \cong \mathcal{O}_{\mathbb{P}^1}(l-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-l-1)$.

($H^0(\mathbb{P}^1, \pi_* L) = H^0(\mathbb{P}^1, \mathcal{O}(l-1)) \cong \mathbb{C}^l$)

As $L \in \mathrm{Pic}^{g-1}(C)$ varies continuously, so should $\pi_* L$.
But if we consider a 1-parameter family of line bundles L_t s.t.

$$\begin{cases} L_t \notin \Theta & (t \neq 0) \\ L_0 \in \Theta & (t = 0) \end{cases}$$

(Here Θ is the Theta divisor of $\mathrm{Pic}^{g-1}(C)$)
$\{ \mathcal{O}_C(P_1 + \cdots + P_{g-1}) \mid P_1, \cdots, P_{g-1} \in C \}$ $\rightarrow h^0(C, L) \neq 0$.

Then

$$\begin{cases} \pi_* L_t \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & (t \neq 0) \\ \pi_* L_0 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & (t = 0) \end{cases}$$

Jumping Phenomenon.

(non-Hausdorff)

Jumping Phenomena cause non-separated structure of the moduli space.

If we want a moduli space of vector bundles having a reasonable topology, we cannot consider all bundles at the same time.

(restrict to "stable" bundles)

<u>Def</u>

E : vector bundle on Σ

The <u>slope $\mu(E)$</u> is defined by

$$\mu(E) := \frac{\deg(E)}{\mathrm{rank}(E)}$$

A bundle E is called <u>stable</u> (resp. <u>semi-stable</u>) if for every subbundle $0 \subsetneq F \subsetneq E$.

$$\mu(F) < \mu(E) \qquad (\text{resp. } \mu(F) \leq \mu(E))$$

<u>Thm</u> (Mumford, Seshadri)

There exist coarse moduli spaces of vector bdls over Σ

$$\mathcal{U}^s_\Sigma(r,d) \subset \mathcal{U}^{ss}_\Sigma(r,d)$$

such that

(1) $\mathcal{U}^{s}_{\Sigma}(r,d)$ is smooth and its points parametrize isomorphic classes of stable bundles of rank r degree d over Σ.
It is an open subset of $\mathcal{U}^{ss}_{\Sigma}(r,d)$.

(2) $\mathcal{U}^{ss}_{\Sigma}(r,d)$ is projective and its points parametrize equivalence classes of semi-stable vector bundles.

$(E_1 \sim E_2 \Leftrightarrow grE_1 \cong grE_2)$

<u>Rem</u> (Moduli Problem / Moduli functor)

Let us consider a functor

$$
\begin{array}{cccc}
\mathcal{M}(-) : & \mathcal{S}ch^{op} & \longrightarrow & \mathcal{S}et \\
 & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
 & S & \longmapsto & \mathcal{M}(S)
\end{array}
$$

s.t. $\mathcal{M}(S) = \{$ equiv. classes of families classified by $S \}$

($\mathcal{S}ch$: category of schemes
$\mathcal{S}et$: category of sets)

cf.

$$
\begin{array}{ccccc}
[E_1] & & [E_2] & \in & \mathcal{M}(S) \\
\downarrow & & \downarrow & & \\
\times s_1 & & \times s_2 & & S \quad (= \mathcal{U}^{s}_{\Sigma}(r,d))
\end{array}
$$

A fine moduli space for $\mathcal{M}(-)$ is a pair (M, Φ) which represents $\mathcal{M}(-)$

i.e.,

$$\Phi : \mathcal{M}(-) \xrightarrow{\sim} \mathrm{Hom}(-, M)$$

A coarse moduli space for $\mathcal{M}(-)$ is a scheme M together with a natural transformation

$$\Phi : \mathcal{M}(-) \longrightarrow \mathrm{Hom}(-, M)$$

s.t.

(1) $\Phi(\{\text{point}\})$ is bijective.

(2) Φ has a universal property

i.e.,

$$\begin{tikzcd}
\mathcal{M}(-) \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \mathrm{Hom}(-, M) \arrow[d, "\exists ! \Omega"] \\
 & \mathrm{Hom}(-, N)
\end{tikzcd}$$

Deformation theory tells us the next lemma

Lem

For $g \geq 2$, $\mathcal{U}^{ss} := \mathcal{U}_{\Sigma}^{ss}(r,d)$, $\mathcal{U}^{s} := \mathcal{U}_{\Sigma}^{s}(r,d)$

(1) $\dim \mathcal{U}^{ss} = 1 + r^2(g-1)$, $\mathcal{U}^s$ is a dense open sub.

(2) Stable bundles are simple

i.e., $H^0(\Sigma, \mathrm{End}(E)) = \mathbb{C}$

(3) Stable bundles are non-singular points of $\mathcal{U}$.

(4) $T_{[E]}\mathcal{U}^s \cong H^1(\Sigma, \mathrm{End}(E))$

$T^*_{[E]}\mathcal{U}^s \cong H^0(\Sigma, \mathrm{End}(E) \otimes \Omega^1_{\Sigma})$

holomorphic 1-form

§2 Hitchin System and Spectral Curves

(Algebro-Geometric version of Arnold-Liouville Integrable systems)

Ref (ACIHS)

An algebraically completely integrable Hamiltonian System consists of a (proper) flat morphism

$$H : M \longrightarrow B$$

where (M, Ψ) is a smooth poisson algebraic variety

B is a smooth variety

s.t. over $B \setminus \Delta$ of some $\Delta \subset B$ proper closed sub.

H is a Lagrangian fibration whose fiberes are isomorphic to abelian varieties

$\approx$ Complex tori

Thm (Hitchin)

part of (?)

$\llcorner$ $T^*\mathcal{U}^s_\Sigma$ supports a natural ACIHS

we call it Hitchin (integrable) system.

The total space of $T^*\mathcal{U}^s_\Sigma$ parametrizes pairs (E, φ) s.t.

(1) E is a stable vec. bdl. $/\Sigma$, $rk = r$, $\deg = d$

(2) $\varphi \in H^1(\Sigma, End(E))^* \cong H^0(\Sigma, End(E) \otimes \Omega^1_\Sigma)$

i.e., 1-form valued endomorphisms of E

$$\varphi : E \longrightarrow E \otimes \Omega^1_\Sigma. \quad (\mathcal{O}_\Sigma\text{-linear})$$

Such a pair is called Higgs bundle and φ ; Higgs field.

The Hamiltonian map of the Hitchin system is the characteristic polynomial map

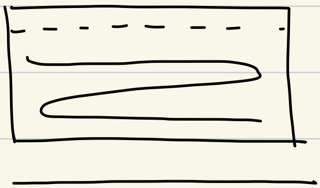
$$H : \quad T^*\mathcal{U}^s_\Sigma(r,d) \longrightarrow B_\omega := \bigoplus_{i=1}^{r} H^0(\Sigma, \Omega^{\otimes i}_\Sigma)$$

$$\qquad (E, \varphi) \longmapsto P := [(b_1, \dots, b_r)]$$

$$P(y) = \mathrm{char}(\varphi)(y) = y^r - \mathrm{tr}(\varphi) y^{r-1} + \cdots + (-1)^r \det(\varphi) = \sum_{i=0}^{r} b_i \cdot y^{r-i}$$

$$b_i := (-1)^i \cdot \mathrm{tr}(\Lambda^i \varphi) \in H^0(\Sigma, \Omega^{\otimes i}_\Sigma), \quad b_0 = 1$$

The inverse image C of the zero section in $\Omega_\Sigma^{\otimes r}$ corresponds to P is called a spectral curve.



$$C \subset \mathrm{Tot}(\Omega^1_\Sigma), \quad L \text{ on } C$$
$$\begin{array}{c} C \\ \downarrow \\ \Sigma \end{array} \qquad \pi_* L = E \text{ on } \Sigma$$

Prop (BNR correspondence)

If C is an irreducible and reduced (integral) spectral curve, there is a bijection between

- $\{ (E, \varphi) \text{ with spectral curve } C \} / \cong$
- $\{ L \,;\, \text{rank } 1 \text{ torsion free sheaf over } C \} / \cong$

(L corresp. to eigen vectors of φ
$c \in C$ corresp. to eigen values of φ)

$$\begin{array}{ccc} H : T^* \mathcal{U}^s_\Sigma & \longrightarrow & B_\omega \\ \cup & & \cup \\ H^{-1}(P) & \longmapsto & \{P\} \end{array}$$

$$H^{-1}(P) \underset{\text{open}}{\subset} \mathrm{Pic}^{d + r(1-g_\Sigma) + g_C - 1}(C)$$

To get whole $\mathrm{Pic}^{\bullet}(C)$, we need to relax the stability condition of E (E, φ).

Def

A pair (E, φ) is stable (resp. semi-stable)
if for every φ-invariant subbundle of E
i.e., $0 \subsetneq F \subsetneq E$ s.t. $\varphi(F) \subset F \otimes \Omega_\Sigma^1$

$\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$).

Thm

There exists a coarse moduli space $\mathrm{Higgs}_\Sigma(r,d)$
parametrizing equiv. classes of semi stable Higgs bundles

$$H : \mathrm{Higgs}_\Sigma(r,d) \longrightarrow \mathcal{B}_\omega$$

is a proper algebraic morphism. (ACIHS)

$(H^{-1}(P) = \mathrm{Pic}^{\circ}(C))$

§3 Non Abelian Hodge Theory

A deep reason for working with the above def. of stability is provided by the following theorems.

Thm (NAH, Simpson)

There is a canonical real analytic diffeomorphism

$$\mathrm{Higgs}_\Sigma(r,0) \cong \mathrm{Hom}^{(\text{semi-simple})}(\pi_1(\Sigma), GL_r(\mathbb{C}))/\sim_{GL_r(\mathbb{C})}$$

$A \sim B \overset{\text{def}}{\iff} \exists Q \in GL_r(\mathbb{C})$ s.t. $B = QAQ^{-1}$.

Thm (Riemann-Hilbert Correspondence)

since $\dim \Sigma = 1$, $\{\nabla\}$ are automatically flat

$\mathrm{Conn}_\Sigma(r)$: moduli space of flat connections $(E,\nabla)/\simeq$

$\nabla : E \longrightarrow E \otimes \Omega^1_\Sigma$ s.t. $\nabla(fs) = df \otimes s + f\nabla s$

Then there exists a complex analytically isomorphism

$$\mathrm{Conn}_\Sigma(r) \cong \mathrm{Hom}^{(s.s)}(\pi_1(\Sigma), GL_r(\mathbb{C}))/\sim$$

$$\underset{\omega}{E_\rho} = (\tilde{\Sigma} \times \mathbb{C}^r)/\pi_1(\Sigma) \longleftarrow\!\shortmid \underset{\psi}{\rho}$$

(which admits natural flat conn.)

We have three moduli spaces :

- Betti moduli space $\quad \mathrm{Hom}^{(s.s)}(\pi_1(\Sigma), GL_r(\mathbb{C}))/\sim$
- de Rham moduli space $\quad \mathrm{Conn}_\Sigma(r)$
- Dolbeault moduli space $\quad \mathrm{Higgs}_\Sigma(r,0)$.

There are other correspondences between them
(Kobayashi-Hitchin, λ-connections, Harmonic bundles, ...)

Why "Non Abelian Hodge" ?? (by simpson)

Hodge theory tells us that for compact Kähler X

$$H^n_{dR}(X,\mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}_{Dol}(X)$$

$$\left(H^{p,q}_{Dol}(X) := H^q(X, \Omega^p_X) \right)$$

For $n=1$, $\quad H^1(X,\mathbb{C}) \cong H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X)$

$\qquad\qquad$ Betti $\qquad\qquad$ Conn. $\qquad\qquad$ Higgs.

Since

$$H^1(X,\mathbb{C}) \cong \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}) \cong \mathrm{Hom}(\pi_1(X),\mathbb{C})$$

replace $\mathbb{C}$ by $GL_n(\mathbb{C})$ (non-abelian gp.)

- $H^1(X,\mathbb{C}) \rightsquigarrow H^1(X, GL_n(\mathbb{C})) \leftrightarrow \mathrm{Hom}(\pi_1(X), GL_n(\mathbb{C}))$
- $H^1(X,\mathcal{O}_X) \rightsquigarrow \check{H}^1(X, GL_n(\mathcal{O}_X))$ $\qquad$ holom. vec. bdl.
- $H^0(X,\Omega^1_X) \rightsquigarrow H^0(X, \underset{(\mathrm{End}(E_\alpha))}{GL_n(\mathbb{C})} \otimes \Omega^1_X)$ $\qquad$ "Higgs bdl."